

The truncated Fourier operator. IV.

Victor Katsnelson Ronny Machluf

Abstract

We consider the formal prolate spheroid differential operator on a finite symmetric interval and describe all its self-adjoint boundary conditions. Only one of these boundary conditions corresponds to a self-operator differential operator which commutes with the Fourier operator truncated on the considered finite symmetric interval.

4 Self-adjoint boundary conditions for the prolate spheroid differential operator.

The study of the spectral theory of the Fourier operator restricted on a finite symmetric interval $[-a, a]$:

$$(\mathcal{F}_E x)(t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{it\xi} x(\xi) d\xi, \quad t \in E, \quad E = [-a, a],$$

$$\mathcal{F}_E : L^2(E) \rightarrow L^2(E), \quad (4.1)$$

is closely related to study of the differential operator generated by the differential expression (or *formal differential operator*) L :

$$(Lx)(t) = -\frac{d}{dt} \left(1 - \frac{t^2}{a^2} \right) \frac{dx(t)}{dt} + t^2 x(t). \quad (4.2)$$

Mathematics Subject Classification: (2000). Primary 47E05, 34E05; Secondary 33E10.

Keywords: Truncated Fourier operator, prolate spheroid differential operator, selfadjoint extensions of singular differential operators, abstract boundary conditions, commuting operators.

The operator L is said to be the **prolate spheroid differential operator**.

The relationship between the spectral theory theories of the integral operator $\mathcal{F}_E^* \mathcal{F}_E$, $E = [-a, a]$, and the prolate spheroid differential operator was discovered in the series of remarkable papers [SlPo], [LaP1], [LaP2], where this relationship has been ingeniously used for developing the spectral theory of the operator $\mathcal{F}_E^* \mathcal{F}_E$. (See also [Sl2], [Sl3].) Actually the reasoning of [SlPo], [LaP1], [LaP2] can be easily applied to the spectral theory of the operator \mathcal{F}_E itself rather the operator $\mathcal{F}_E^* \mathcal{F}_E$.

It should be emphasized that what was used in [SlPo], [LaP1], [LaP2] this is a certain system of eigenfunctions related to the differential expression L , (4.2). These eigenfunctions are known as *prolate wave functions*. The prolate wave functions themselves were used much before the series of the papers [SlPo], [LaP1], [LaP2] was published. These functions naturally appears by separation of variables in the Laplace equation in spheroidal coordinates. However this was the work [SlPo], [LaP1], [LaP2] where the prolate functions were first used for solving the spectral problem related to the Fourier analysis on a finite symmetric interval. Until now, there is no clear understanding why the approach used in [SlPo], [LaP1], [LaP2] works. This is a a lucky accident which still waits for its explanation. (See [Sl3].)

Actually eigenfunctions are related not to the the differential expression itself but to a certain differential operator generated by the differential expression. This differential operator is generated not only by the differential expression but also by certain boundary condition. In the case $E = (-\infty, \infty)$, the differential operator generated by the differential expression $-\frac{d^2}{dt^2} + t^2$ on the class smooth finite functions (or the class of smooth fast decaying functions) is essentially selfadjoint: the closure of this operator is a selfadjoint operator. Thus in the case $E = (-\infty, \infty)$ there is no need to discuss the boundary condition because there is no such boundary conditions.

In contrast to the case $E = (-\infty, \infty)$, in the case $E = [-a, a]$, $0 < a < \infty$ the minimal differential operator $-\frac{d}{dt} \left(1 - \frac{t^2}{a^2} \right) \frac{dx}{dt} + t^2$ is symmetric but is *not* self-adjoint. This minimal operator admits the family of self-adjoint extensions. Each of this selfadjoint extensions is described by a certain boundary conditions at the end points of the interval $[-a, a]$. The set of all such extensions can be parameterized by the set of all 2×2 unitary matrices.

It turns out that only one of these extensions commutes with the

truncated Fourier operator \mathcal{F}_E , $E = [-a, a]$. To our best knowledge, until now no attention was paid to this aspect. In the present paper, we in particular investigate the question which extensions of the minimal differential operator generated by L , (4.2), commute with L .

Analysis of solutions of the equation $Lx = \lambda x$ near singular points.

For the differential equation

$$-\frac{d}{dt}\left(1 - \frac{t^2}{a^2}\right)\frac{dx(t)}{dt} + t^2x(t) = \lambda x(t), \quad t \in \mathbb{C}, \quad (4.3)$$

considered in complex plane, the points $-a$ and a are the regular singular point. Let us investigate the asymptotic behavior of solutions of this equation near these points. (Actually we need to know this behavior only for real $t \in (-a, a)$, but it is much easier to investigate this question using some knowledge from the analytic theory of differential equation.) Concerning the analytic theory of differential equation see [Sm, Chapter 5].

Let us outline an analysis of solution of the equation near the point $t = -a$. Change of variable

$$t = -a + s, \quad x(-a + s) = y(s)$$

reduces the equation (4.3) to the form

$$s\frac{d^2y(s)}{ds^2} + p(s)\frac{dy(s)}{ds} + q(s)y(s) = 0, \quad (4.4)$$

where $p(s)$ and $q(s)$ are functions holomorphic within the disc $|s| < 2a$, moreover $p(0) = 1$:

$$p(s) = 1 + \sum_{k=1}^{\infty} p_k s^k, \quad q(s) = \sum_{k=0}^{\infty} q_k(\lambda) s^k. \quad (4.5)$$

An explicit calculation with power series give:

$$p_1 = -\frac{1}{2a}; \quad q_0 = \frac{\lambda a}{2} - \frac{a^3}{2}, \quad q_1 = \frac{\lambda}{4} + \frac{3}{4}a^2. \quad (4.6)$$

Now we turn to the analytic theory of differential equations. The results of this theory which we need are presented for example in [Sm, Chapter 5], see especially section **98** there. We seek the solution of the equation (4.4)-(4.5) in the form

$$y(s) = s^\rho \sum_{k=0}^{\infty} c_k s^k.$$

Substituting this to the left-hand side of the equation (4.4)-(4.5) and equating the coefficients of like powers of s to zero we obtain the equations for the determination of ρ and c_k . In particular, the equation corresponding to the power s^0 is of the form:

$$c_0 \rho^2 = 0.$$

The coefficient c_0 plays the role of a normalizing constant, and we may take

$$c_0 = 1. \quad (4.7)$$

Equation for ρ , the so called *characteristic equation*, is of the form

$$\rho^2 = 0. \quad (4.8)$$

This equation has the root $\rho = 0$, and this root is multiple. According to general theory, the equation (4.4)-(4.5) has two solutions $y_1(s)$ and $y_2(s)$ possessing the properties:

The solution $y_1(s)$ is a function holomorphic in the disc $|s| < 2a$ satisfying the normalizing condition $y_1(0) = 1$. The solution $y_2(s)$ is of the form $y_2(s) = y_1(s) \ln s + z(s)$, where $z(s)$ is a function holomorphic in the disc $|s| < 2a$ and satisfying the condition $z(0) = 0$. We may calculate explicitly several first coefficients of power expansions

$$y_1(s) = 1 + \sum_{k=1}^{\infty} c_k s^k, \quad z(s) = \sum_{k=1}^{\infty} d_k s^k :$$

$$c_1 = \frac{a^3}{2} - \frac{\lambda a}{2}, \quad d_1 = \lambda a - a^3 + \frac{1}{2a}.$$

Returning to the variable $t = -a + s$, we get the following result:

Lemma 4.1. *Let L be the differential expression defined by (4.2), and $\lambda \in \mathbb{C}$ be arbitrary fixed.*

1. *There exist two solutions $x_1^-(t, \lambda)$ and $x_2^-(t, \lambda)$ of the equation $Lx(t) = \lambda x(t)$ possessing the properties:*
 - a. *The function $x_1^-(t, \lambda)$ is holomorphic in the disc $|t + a| < 2a$, and satisfy the normalizing condition $x_1^-(-a, \lambda) = 1$;*
 - b. *The function $x_2^-(t, \lambda)$ is of the form*

$$x_2^-(t, \lambda) = x_1^-(t, \lambda) \ln(t + a) + w^-(t, \lambda),$$

where the function $w^-(t, \lambda)$ is holomorphic in the disc $|t + a| < 2a$ and satisfy the condition $w^-(-a, \lambda) = 0$.

2. There exist two solutions $x_1^+(t, \lambda)$ and $x_2^+(t, \lambda)$ of the equation $Lx(t) = \lambda x(t)$ possessing the properties:
- a. The function $x_1^+(t, \lambda)$ is holomorphic in the disc $|t - a| < 2a$, and satisfy the normalizing condition $x_1^+(a, \lambda) = 1$;
 - b. The function $x_2^+(t, \lambda)$ is of the form

$$x_2^+(t, \lambda) = x_1^+(t, \lambda) \ln(a - t) + w^+(t, \lambda),$$

where the function $w^+(t, \lambda)$ is holomorphic in the disc $|t + a| < 2a$ and satisfy the condition $w^+(a, \lambda) = 0$.

Given a fixed λ , the solutions $x_1^-(t, \lambda)$, $x_2^-(t, \lambda)$ are linearly independent, therefore arbitrary solution $x(t, \lambda)$ of the equation (4.3) can be expanded into a linear combination

$$x(t, \lambda) = c_1^- x_1^-(t, \lambda) + c_2^- x_2^-(t, \lambda). \quad (4.9a)$$

The solutions $x_1^+(t, \lambda)$, $x_2^+(t, \lambda)$ also are linearly independent, and the solution $x(t, \lambda)$ can be also expanded into the other linear combination

$$x(t, \lambda) = c_1^+ x_1^+(t, \lambda) + c_2^+ x_2^+(t, \lambda). \quad (4.9b)$$

Here c_1^\pm, c_2^\pm are constants (with respect to t). The solution $x_1^-(t, \lambda)$ is bounded and the solution $x_2^-(t, \lambda)$ grows logarithmically as $t \rightarrow -a$. Therefore the solution $x(t, \lambda)$ is square integrable near the point $t = -a$. For the same reason, the the solution $x(t, \lambda)$ is square integrable near the point $t = a$. Thus we prove the following result.

Lemma 4.2. *Given $\lambda \in \mathbb{C}$, then every solution $x(t, \lambda)$ of the equation (4.3) satisfy the condition*

$$\int_{-a}^a |x(t, \lambda)|^2 dt < \infty. \quad (4.10)$$

Differential operators related to the differential expression L , (4.11).

With the differential expression (or, in other words, the *formal differential operator*) L ,

$$L = -\frac{d}{dt} \left(1 - \frac{t^2}{a^2} \right) \frac{d}{dt} + t^2, \quad (4.11)$$

various differential operators may be related according to whether boundary conditions are posed on functions from their domains of definition.

Definition 4.1. The set \mathcal{A} is the set of complex-valued functions $x(t)$ defined on the open interval $(-a, a)$ and satisfied the following conditions:

1. The derivative $\frac{dx(t)}{dt}$ of the function $x(t)$ exists at every point t of the interval $(-a, a)$;
2. The function $\frac{dx(t)}{dt}$ is absolutely continuous on every compact subinterval of the interval $(-a, a)$;

Definition 4.2. The set $\mathring{\mathcal{A}}$ is the set of complex-valued functions $x(t)$ defined on the open interval $(-a, a)$ and satisfied the following conditions:

1. The function $x(t)$ belongs to the set \mathcal{A} defined above;
2. The support $\text{supp } x$ of the function $x(t)$ is a compact subset of the open interval $(-a, a)$: $(\text{supp } x) \Subset (-a, a)$.

Definition 4.3. The differential operator \mathcal{L}_{\max} is defined as follows:

1. The domain of definition $\mathcal{D}_{\mathcal{L}_{\max}}$ of the operator \mathcal{L}_{\max} is:

$$\mathcal{D}_{\mathcal{L}_{\max}} = \{x : x(t) \in L^2((-a, a)) \cap \mathcal{A} \text{ and } (Lx)(t) \in L^2((-a, a))\}, \quad (4.12a)$$

where $(Lx)(t)$ is defined¹ by (4.2).

2. The action of the operator \mathcal{L}_{\max} is:

$$\text{For } x \in \mathcal{D}_{\mathcal{L}_{\max}}, \quad \mathcal{L}_{\max} x = Lx. \quad (4.12b)$$

The operator \mathcal{L}_{\max} is said to be the maximal differential operator generated by the differential expression L , (4.11).

The minimal differential operator \mathcal{L}_{\min} is the restriction of the maximal differential operator \mathcal{L}_{\max} on the set of functions which is some sense vanish at the endpoint of the interval $(-a, a)$. The precise definition is presented below.

Definition 4.4. The operator \mathcal{L}_{\min} is the closure² of the operator $\mathring{\mathcal{L}}$:

$$\mathcal{L}_{\min} = \text{clos}(\mathring{\mathcal{L}}), \quad (4.13a)$$

where the operator $\mathring{\mathcal{L}}$ is the restriction of the operator \mathcal{L}_{\max} :

$$\mathring{\mathcal{L}} \subset \mathcal{L}_{\max}, \quad \mathring{\mathcal{L}} = \mathcal{L}_{\max}|_{\mathcal{D}_{\mathring{\mathcal{L}}}}, \quad \mathcal{D}_{\mathring{\mathcal{L}}} = \mathcal{D}_{\mathcal{L}_{\max}} \cap \mathring{\mathcal{A}}. \quad (4.13b)$$

¹Since $x \in \mathcal{A}$, the expression $(Lx)(t)$ is well defined.

²Since the operator $\mathring{\mathcal{L}}$ is symmetric, it is closable.

By \langle , \rangle we denote the standard scalar product in $L^2((-a, a))$:

$$\text{For } u, v \in L^2((-a, a)), \quad \langle u, v \rangle = \int_{-a}^a u(t) \overline{v(t)} dt.$$

The properties of the operators \mathcal{L}_{\min} and \mathcal{L}_{\max} :

1. The operator \mathcal{L}_{\min} is symmetric:

$$\langle \mathcal{L}_{\min} x, y \rangle = \langle x, \mathcal{L}_{\min} y \rangle, \quad \forall x, y \in \mathcal{D}_{\mathcal{L}_{\min}}; \quad (4.14)$$

In other words, the operator \mathcal{L}_{\min} is contained in its adjoint:

$$\mathcal{L}_{\min} \subseteq (\mathcal{L}_{\min})^*;$$

2. The operators \mathcal{L}_{\min} and \mathcal{L}_{\max} are mutually adjoint:

$$(\mathcal{L}_{\min})^* = \mathcal{L}_{\max}, \quad (\mathcal{L}_{\max})^* = \mathcal{L}_{\min}; \quad (4.15)$$

In 1930 John von Neumann, [Neu], has found a criterion for the existence of a self-adjoint extension of a symmetric operator A_0 and has described all such extensions. This criterion is formulated in terms of deficiency indices of the symmetric operator.

Definition 4.5. Let A_0 be an operator in a Hilbert space \mathfrak{H} . We assume that the domain of definition \mathcal{D}_{A_0} is dense in \mathfrak{H} and that the operator A_0 is symmetric, that is³

$$A_0 \subseteq (A_0)^*. \quad (4.16)$$

For complex number λ , consider the subspace

$$\mathcal{N}_\lambda = \mathfrak{H} \ominus ((A_0 - \lambda I)\mathcal{D}_{A_0}), \quad (4.17a)$$

or, what is equivalent,

$$\mathcal{N}_\lambda = \{x \in \mathfrak{H} : (A_0)^*x = \bar{\lambda}x\}. \quad (4.18)$$

The dimension $\dim \mathcal{N}_\lambda$ is constant in the upper half-plane $\text{Im } \lambda > 0$ and in the lower half-plane $\text{Im } \lambda < 0$:

$$\dim \mathcal{N}_\lambda = n_+, \quad \text{Im } \lambda > 0, \quad (4.19a)$$

$$\dim \mathcal{N}_\lambda = n_-, \quad \text{Im } \lambda < 0. \quad (4.19b)$$

³The relation (4.16) means that $\mathcal{D}_{A_0} \subseteq \mathcal{D}_{A_0^*}$ and $A_0x = A_0^*x \forall x \in \mathcal{D}_{A_0}$.

The numbers n^+ and n^- are said to be the deficiency indices of the operator A_0 , and the subspace \mathcal{N}_λ is said to be the deficiency subspace corresponding to the value λ .

Theorem (von Neumann).

1. *The densely defined symmetric operator admits selfadjoint extensions if and only if its deficiency indices are equal:*

$$n_+ = n_- . \quad (4.20)$$

2. *Assume that a symmetric operator A_0 is closed and its deficiency indices are equal. Choose a pair of non-real conjugated complex numbers, for example $\lambda = i$, $\bar{\lambda} = -i$. The set of all selfadjoint extensions of the operator A is in one-to-one correspondence with the set of all unitary operators acting from the deficiency subspace \mathcal{N}_i into the deficiency subspace \mathcal{N}_{-i} . In particular, if $n_+ = n_- = 0$, the operator A_0 already is selfadjoint.*

We apply the von Neumann Theorem to the situation where the operator \mathcal{L}_{\min} is taken as the operator A_0 . Then the equation

$$(A_0)^* x = \lambda x$$

takes the form

$$\mathcal{L}_{\max} x = \bar{\lambda} x ,$$

that is the differential equation

$$-\frac{d}{dt} \left(1 - \frac{t^2}{a^2} \right) \frac{dx(t)}{dt} + t^2 x(t) = \bar{\lambda} x(t), \quad t \in (-a, a), \quad (4.21)$$

under the extra condition $x(t) \in L^2(-a, a)$. In particular, the dimension of the deficiency space \mathcal{N}_λ coincides with the dimension of the linear space of the set of solutions of the equation (4.21) belonging to $L^2(-a, a)$. According to Lemma 4.2, every solution of the equation (4.21) belongs to $L^2(-a, a)$. Thus we prove the following

Lemma 4.3. *For the operator \mathcal{L}_{\min} , the deficiency indices are:*

$$n_+(\mathcal{L}_{\min}) = 2, \quad n_-(\mathcal{L}_{\min}) = 2 . \quad (4.22)$$

Thus, the operator \mathcal{L}_{\min} is symmetric, but not selfadjoint. The set of all its selfadjoint extensions can be parameterized by the set of all 2×2 unitary operators acting from the two-dimensional Hilbert space

\mathcal{N}_i onto two-dimensional Hilbert space \mathcal{N}_{-i} , where $\mathcal{N}_{\pm i}$ are defect subspaces of the operator \mathcal{L}_{\min} .

Selfadjoint extensions of operators and self-orthogonal subspaces.

J. von Neumann, [Neu], reduced the construction of a selfadjoint extension for a symmetric operator A_0 to an equivalent problem of construction of an unitary extension of an appropriate isometric operator - the Caley transform of this symmetric operator. This approach was also developed by M. Stone, [St], and then used by many others.

In some situations, it is much more convenient to use the construction of extensions based on the so called *boundary forms*. Especially convenient is the usage of this construction for differential operators. The first version of the extension theory based on abstract symmetric boundary conditions, was developed by J.W. Calkin, [Cal]. Subsequently various versions of the extension theory of symmetric operators in terms of abstract boundary conditions were developed. The dual problem of the descriptions of extensions of symmetric boundary relations was also considered. See [RoB], [Koch], [Br].

Considering the symmetric operator A_0 (4.16) acting in a Hilbert space \mathfrak{H} , we introduce the bilinear form Ω

$$\Omega(x, y) = \frac{\langle A_0^* x, y \rangle - \langle x, A_0^* y \rangle}{i}, \quad \Omega : \mathcal{D}_{A_0^*} \times \mathcal{D}_{A_0^*} \rightarrow \mathbb{C}. \quad (4.23a)$$

The bilinear form Ω is hermitian:

$$\Omega(x, y) = \overline{\Omega(y, x)}, \quad \forall x, y \in \mathcal{D}_{A_0^*},$$

and possesses the property

$$\Omega(x, y) = 0, \quad \forall x \in \mathcal{D}_{A_0^*}, y \in \mathcal{D}_{A_0}.$$

This property allows to consider the form Ω as a form on the factor-space \mathcal{E} :

$$\mathcal{E} = \mathcal{D}_{A_0^*} / \mathcal{D}_{A_0}. \quad (4.24)$$

We use the same notation for the form induced on the factor space \mathcal{E} :

$$\Omega(x, y) = \frac{\langle A^* x, y \rangle - \langle x, A^* y \rangle}{i}, \quad \Omega : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{C}. \quad (4.23b)$$

Definition 4.6. *The form Ω , (4.23), is said to be the boundary form. The factor space \mathcal{E} is said to be the boundary space.*

It turns out that

$$\dim \mathcal{E} = n_+ + n_- , \quad (4.25)$$

where n_+ and n_- are deficiency indices of the operator A_0 , and that *the form Ω is not degenerate on \mathcal{E}* . The non-degeneracy of the form means:

$$\text{For every non-zero } x \in \mathcal{E}, \text{ there exists } y \in \mathcal{E} \text{ such that } \Omega(x, y) \neq 0. \quad (4.26)$$

Let \mathcal{S} be a subspace of the factor space \mathcal{E} :

$$\mathcal{S} \subseteq \mathcal{E}. \quad (4.27a)$$

We identify \mathcal{S} and its preimage with respect to the factor-mapping $\mathcal{D}_{A_0^*} \rightarrow \mathcal{D}_{A_0^*}/\mathcal{D}_{A_0} (= \mathcal{E})$ and use the same notation \mathcal{S} for a subspace in \mathcal{E} and its preimage in \mathcal{D}_{A_0} :

$$\mathcal{D}_{A_0} \subseteq \mathcal{S} \subseteq \mathcal{D}_{A_0^*}. \quad (4.27b)$$

To every \mathcal{S} satisfying (4.27b), an extension of the operator A_0 is related. We denote this extension by $A_{\mathcal{S}}$:

$$\mathcal{D}_{A_{\mathcal{S}}} = \mathcal{S}, \quad A_0 \subseteq A_{\mathcal{S}} \subseteq A_0^*.$$

The operator $(A_{\mathcal{S}})^*$, which is the operator adjoint to the operator $A_{\mathcal{S}}$, is related to the subspace \mathcal{S}^{\perp} :

$$(A_{\mathcal{S}})^* = A_{\mathcal{S}^{\perp}}, \quad (4.28)$$

where $\mathcal{S}^{\perp\Omega}$ is the *orthogonal complement* of the subspace \mathcal{S} with respect to the hermitian form Ω :

$$\mathcal{S}^{\perp\Omega} = \{x \in \mathcal{E} : \Omega(x, y) = 0 \quad \forall y \in \mathcal{S}\}. \quad (4.29)$$

In particular we prove the following result:

Lemma 4.4. *The extension $A_{\mathcal{S}}$ of the symmetric operator A_0 is a selfadjoint operator: $A_{\mathcal{S}} = (A_{\mathcal{S}})^*$, if and only if the subspace \mathcal{S} which appears in (4.27b) possesses the property:*

$$\mathcal{S} = \mathcal{S}^{\perp\Omega}. \quad (4.30)$$

Definition 4.7. *The subspace \mathcal{S} of the space \mathcal{E} is said to be Ω -self-complementary if it possess the property (4.30).*

It turns out that self-complementary subspaces exist if and only if the form Ω , (4.23b), has equal numbers of positive and negative squares. (Which condition is equivalent to the condition $n_+ = n_-$.)

Thus, *the problem of description of all self-adjoint extensions of a symmetric operator A_0 can be reformulated as the problem of description of subspaces of the space \mathcal{E} , (4.24), which are self-complementary with respect to the (non-degenerated) boundary form Ω , (4.23b).*

Selfadjoint extensions of symmetric differential operators. The description of selfadjoint extensions of a symmetric operator A_0 becomes especially transparent in the case when this symmetric operator is formally selfadjoint ordinary differential operator, regular or singular. In this case the *hermitian form* Ω , (4.23a), can be expressed in terms of *boundary conditions* of functions from domain of definitions of the operator A_0^* . This justifies the terminology introduced in Definition 4.6.

We illustrate the situation as applied to the case where the symmetric operator A_0 is the minimal differential operator \mathcal{L}_{\min} generated by the formal prolate spheroid differential operator L . Then the adjoint operator A_0^* is the maximal differential operator \mathcal{L}_{\max} (See Definitions 4.4 and 4.3.) The problem of description of selfadjoint differential operators generated by a given formal differential operator, has the long history. See, for example, [Kr], [Nai, Chapter 5]. The book of [DuSch] is the storage of wisdom in various aspects of the operator theory, in particular is self-adjoint ordinary differential operators. See especially Chapter XIII of [DuSch].

In principle we may incorporate the question of description of self-adjoint boundary condition for the prolate spheroid differential operators in one or other of the existing abstract schemes which is devoted to such a description in one or other generality. However to adopt our question to such a scheme one needs to agree the notation, the terminology, etc. This auxiliary work may obscure the presentation. To make the presentation more transparent, we prefer to act independently on the existing general considerations and to develop what we need from the blank page.

We use the notations

$$p(t) = 1 - \frac{t^2}{a^2}, \quad q(t) = t^2, \quad -a < t < a.$$

In this notation, the formal differential operator L introduced in (4.11)

is:

$$(Lx)(t) = -\frac{d}{dt}\left(p(t)\frac{dx(t)}{dt}\right) + q(t)x(t), \quad -a < t < a.$$

For every $x, y \in \mathcal{A}$,

$$(Lx(t))\overline{y(t)} - x(t)\overline{(Ly(t))} = \frac{d}{dt}[x(t), y(t)], \quad -a < t < a,$$

where

$$[x(t), y(t)] = -p(t)\left(\frac{dx(t)}{dt}\overline{y(t)} - x(t)\frac{dy(t)}{dt}\right).$$

Therefore, for every $x, y \in \mathcal{A}$ and for every $\alpha, \beta : -a < \alpha < \beta < a$,

$$\int_{\alpha}^{\beta} \left((Lx(t))\overline{y(t)} - x(t)\overline{(Ly(t))} \right) dt = [x, y](\beta) - [x, y](\alpha). \quad (4.31)$$

Lemma 4.5. *For every $x, y \in \mathcal{D}_{\mathcal{L}_{\max}}$, there exist the limits*

$$[x, y]_{-a} \stackrel{\text{def}}{=} \lim_{\alpha \rightarrow -a+0} [x, y](\alpha), \quad [x, y]^a \stackrel{\text{def}}{=} \lim_{\beta \rightarrow a-0} [x, y](\beta). \quad (4.32a)$$

Proof. Since $x(t), y(t), (Lx(t)), (Ly(t))$ belongs to $L^2((-a, a))$, then $\int_{-a}^a \left| ((Lx(t))\overline{y(t)} - x(t)\overline{(Ly(t))}) \right| dt < \infty$. Therefore

$$\begin{aligned} \int_{-a}^a \left(((Lx(t))\overline{y(t)} - x(t)\overline{(Ly(t))}) \right) dt &= \\ &= \lim_{\substack{\alpha \rightarrow -a+0 \\ \beta \rightarrow a-0}} \int_{\alpha}^{\beta} \left(((Lx(t))\overline{y(t)} - x(t)\overline{(Ly(t))}) \right) dt. \end{aligned}$$

Concerning this and related result see for example [HuPy, Chapter 10]. \square

The boundary form Ω , constructed from the operator $A_0 = \mathcal{L}_{\min}$ according to (4.23a),

$$\Omega_L(x, y) = \frac{\langle \mathcal{L}_{\max} x, y \rangle - \langle x, \mathcal{L}_{\max} y \rangle}{i}$$

can be expressed in the term of the generalized boundary values:

$$\Omega_L(x, y) = \frac{[x, y]^a - [x, y]_{-a}}{i}. \quad (4.32b)$$

According to (4.25) and Lemma 4.3, the dimension of the boundary space \mathcal{E}_L : $\mathcal{E}_L = \mathcal{D}_{\mathcal{L}_{\max}}/\mathcal{D}_{\mathcal{L}_{\min}}$ is:

$$\dim \mathcal{E}_L = 4. \quad (4.33)$$

To make calculation explicit, we choose a special basis in the space \mathcal{E}_L in which the bilinear form Ω_L is reduced to "sum of squares". The asymptotic behavior of solutions of the equation $Lx = 0$ near the endpoints of the interval $(-a, a)$, described in Lemma 4.1, prompts us the choice of such a basis. Let $\varphi_-(t), \psi_-(t), \varphi_+(t), \psi_+(t)$ be smooth functions such that

$$\begin{aligned} \varphi_-(t) &= 1, & -a < t < -a/2, & \quad \varphi_-(t) = 0, & \quad a/2 < t < a, \\ \psi_-(t) &= \ln(a+t), & -a < t < -a/2, & \quad \psi_-(t) = 0, & \quad a/2 < t < a, \\ \varphi_+(t) &= 0, & -a < t < -a/2, & \quad \varphi_+(t) = 1, & \quad a/2 < t < a, \\ \psi_+(t) &= 0, & -a < t < -a/2, & \quad \psi_+(t) = \ln(a-t), & \quad a/2 < t < a. \end{aligned} \quad (4.34)$$

It is cleat that if χ is an arbitrary smooth real valued function, then $\Omega_L(\chi, \chi) = 0$. In particular,

$$\Omega_L(\chi, \chi) = 0, \quad \text{if } \chi \text{ is one of the functions } \varphi_-, \psi_-, \varphi_+, \psi_+. \quad (4.35a)$$

It is clear that

$$\Omega_L(\chi_-, \chi_+) = 0, \quad \text{if } \chi_{\pm} \text{ is one of the functions } \varphi_{\pm}, \psi_{\pm}. \quad (4.35b)$$

Direct calculation shows that

$$\Omega_L(\varphi_-, \psi_-) = -\frac{2}{a}, \quad \Omega_L(\varphi_+, \psi_+) = -\frac{2}{a}. \quad (4.35c)$$

Thus, the Gram matrix (with respect to the hermitian form Ω_L) of the vectors $\varphi_-, \psi_-, \varphi_+, \psi_+$ is:

$$\frac{a}{2} \cdot \begin{bmatrix} \Omega_L(\varphi_-, \varphi_-) & \Omega_L(\varphi_-, \psi_-) & \Omega_L(\varphi_-, \varphi_+) & \Omega_L(\varphi_-, \psi_+) \\ \Omega_L(\psi_-, \varphi_-) & \Omega_L(\psi_-, \psi_-) & \Omega_L(\psi_-, \varphi_+) & \Omega_L(\psi_-, \psi_+) \\ \Omega_L(\varphi_+, \varphi_-) & \Omega_L(\varphi_+, \psi_-) & \Omega_L(\varphi_+, \varphi_+) & \Omega_L(\varphi_+, \psi_+) \\ \Omega_L(\psi_+, \varphi_-) & \Omega_L(\psi_+, \psi_-) & \Omega_L(\psi_+, \varphi_+) & \Omega_L(\psi_+, \psi_+) \end{bmatrix} = J, \quad (4.36)$$

where

$$J = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}. \quad (4.37)$$

The rank of the Gram matrix is equal to the dimension of the space \mathcal{E}_L :

$$\text{rank } J = \dim \mathcal{E}_L = 4.$$

Therefore, the vectors φ_- , ψ_- , φ_+ , ψ_+ generate the space $\mathcal{E}_L = \mathcal{D}_{\mathcal{L}_{\max}}/\mathcal{D}_{\mathcal{L}_{\min}}$. In particular, the domain of definition $\mathcal{D}_{\mathcal{L}_{\min}}$ of the minimal differential operator \mathcal{L}_{\min} can be characterized by means of the boundary conditions:

$$\begin{aligned} \mathcal{D}_{\mathcal{L}_{\min}} &= \{x(t) : x(t) \in \mathcal{D}_{\mathcal{L}_{\max}}, \text{ and} \\ &[x, \varphi_-]_{-a} = 0, [x, \psi_-]_{-a} = 0, [x, \varphi_+]^a = 0, [x, \psi_+]^a = 0\}, \end{aligned} \quad (4.38)$$

where the forms $[\cdot, \cdot]_{-a}$, $[\cdot, \cdot]^a$ were introduced in (4.5).

Lemma 4.6. *Let Ω_L be a bilinear form in the space \mathcal{E} defined by (4.5), and J be the matrix (4.37).*

The vector $x^1 = \alpha_-^1 \varphi_- + \beta_-^1 \psi_- + \alpha_+^1 \varphi_+ + \beta_+^1 \psi_+ \in \mathcal{E}_L$ is Ω_L -orthogonal to the vector $x^2 = \alpha_-^2 \varphi_- + \beta_-^2 \psi_- + \alpha_+^2 \varphi_+ + \beta_+^2 \psi_+ \in \mathcal{E}_L$, that is

$$\Omega_L(x^1, x^2) = 0, \quad (4.39a)$$

if and only if the vector-row $v_{x^1} = [\alpha_-^1, \beta_-^1, \alpha_+^1, \beta_+^1] \in \mathcal{V}$ is J -orthogonal to the vector-row $v_{x^2} = [\alpha_-^2, \beta_-^2, \alpha_+^2, \beta_+^2] \in \mathcal{V}$, that is

$$v_{x^1} J v_{x^2}^* = 0, \quad (4.39b)$$

where \mathcal{V} is the space \mathbb{C}^4 of vector-rows equipped by the standard hermitian metric, and the star $$ is the Hermitian conjugation.*

Thus, the problem of description of self-complementary extensions of the operator \mathcal{L}_{\min} is equivalent to the problem of description of Ω_L -self-complementary⁴ subspaces in \mathcal{E} , which in its turn is equivalent to the problem of description of J -self-complementary subspaces in

⁴As soon as the notion of J -orthogonality of two vectors is introduced, (4.39b), the notions of J -orthogonal complement and J -self-orthogonal subspaces can be introduced as well.

\mathbb{C}^4 . The last problem is a problem of the indefinite linear algebra and admits an explicit solutions. We set

$$P_+ = \frac{1}{2}(I + J), \quad P_- = \frac{1}{2}(I - J), \quad (4.40a)$$

More explicitly,

$$P_+ = \frac{1}{2} \begin{bmatrix} 1 & i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \end{bmatrix}, \quad P_- = \frac{1}{2} \begin{bmatrix} 1 & -i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \end{bmatrix}. \quad (4.40b)$$

The matrix J , (4.37), possesses the properties

$$J = J^*, \quad J^2 = I.$$

Therefore the matrices P_+ , P_- , (4.40a), possess the properties

$$P_+^2 = P_+, \quad P_-^2 = P_-, \quad P_+ = P_+^*, \quad P_- = P_-^*, \quad (4.41)$$

$$P_+ P_- = 0, \quad P_+ + P_- = I. \quad (4.42)$$

In other words, the matrices P_+ , P_- are orthogonal projector matrices. These matrices project the space \mathcal{V} onto subspaces \mathcal{V}_+ and \mathcal{V}_- :

$$\mathcal{V}_+ = \mathcal{V}P_+, \quad \mathcal{V}_- = \mathcal{V}P_-. \quad (4.43)$$

These subspaces are orthogonally complementary:

$$\mathcal{V}_+ \oplus \mathcal{V}_- = \mathcal{V}. \quad (4.44)$$

The vector rows

$$e_+^1 = [1, \quad i, 0, 0], \quad e_+^2 = [0, 0, 1, \quad i] \quad (4.45a)$$

and

$$e_-^1 = [1, -i, 0, 0], \quad e_-^2 = [0, 0, 1, -i] \quad (4.45b)$$

form orthogonal⁵ bases in \mathcal{V}_+ and \mathcal{V}_- respectively.

It turns out that J -self-orthogonal subspaces of the space \mathcal{V} are in one-to-one correspondence with unitary operators acting from \mathcal{V}_+ onto \mathcal{V}_- .

⁵ In the standard scalar product on $\mathcal{V} = \mathbb{C}^4$.

Definition 4.8. Let U be an unitary operator acting from \mathcal{V}_+ onto \mathcal{V}_- . As the vector-row v runs over the whole subspace \mathcal{V}_+ , the vector $v + vU$ runs over a subspace of the space \mathcal{V} . This subspace is denoted by \mathcal{S}_U :

$$\mathcal{S}_U = \{v + vU\}, \quad \text{where } v \text{ runs over the whole } \mathcal{V}_+. \quad (4.46)$$

Lemma 4.7.

1. Let U be an unitary operator acting from \mathcal{V}_+ onto \mathcal{V}_- . Then the subspace \mathcal{S}_U is J -self-complementary, that is

$$\mathcal{S}_U = \mathcal{S}_U^{\perp J}.$$

2. Every J -self-complementary subspace \mathcal{S} of the space \mathcal{V} is of the form \mathcal{S}_U :

$$\mathcal{S} = \mathcal{S}_U$$

for some unitary operator $U : \mathcal{V}_+ \rightarrow \mathcal{V}_-$.

3. The correspondence between J -self-complementary subspaces and unitary operators acting from \mathcal{V}_+ onto \mathcal{V}_- is one-to-one;

$$(U_1 = U_2) \Leftrightarrow (\mathcal{S}_{U_1} = \mathcal{S}_{U_2}).$$

Proof. 1. The mapping $v \rightarrow v + vU$ is one-to-one mapping from \mathcal{V}_+ onto \mathcal{S}_U . Indeed, this mapping is surjective by definition of the subspace \mathcal{S}_U . This mapping is also injective. The equality $v + vU = 0$ implies that $v = vU = 0$ since ⁶ $v \perp vU$. In particular, $\dim \mathcal{S}_U = \dim \mathcal{V}_+ (= 2)$.

If v_1 and v_2 are two arbitrary vectors from \mathcal{V}_+ , then the vectors $w_1 = v_1 + v_1U$ and $w_2 = v_2 + v_2U$ are J -orthogonal: $w_1 J w_2^* = 0$. Indeed, since $J = P_+ - P_-$ and $v_k = v_k P_+, v_k U = v_k U P_-$, $k = 1, 2$, then, using the properties (4.41) of P_+ and P_- , we obtain

$$\begin{aligned} w_1 J w_2^* &= (v_1^1 P_+ + v_1^1 U P_-)(P_+ - P_-)(P_+^* v_2^* + P_-^* U^* v_2^*) = \\ &= v_1 v_2^* - v_1 U U^* v_2^*. \end{aligned}$$

Since the unitary operator U preserves the scalar product, then $v_1 v_2^* = v_1 U U^* v_2^*$, hence $w_1 J w_2^* = 0$. Thus, $\mathcal{S}_U \subseteq (\mathcal{S}_U)^{\perp J}$. (The symbol \perp_J means J -orthogonal complement.) Since the Hermitian form $(v_1, v_2) \rightarrow v_1 J v_2^*$ is non-degenerate on \mathcal{V} , then $\dim(\mathcal{S}_U^{\perp J}) = \dim \mathcal{V} - \dim \mathcal{S}_U$. Because $\dim \mathcal{V} - \dim \mathcal{S}_U = \dim \mathcal{S}_U$, we have $\dim \mathcal{S}_U = \dim(\mathcal{S}_U^{\perp J})$. Hence, $\mathcal{S}_U = (\mathcal{S}_U)^{\perp J}$, i.e. the subspace \mathcal{S}_U is J -self-complementary.

⁶Recall that $v \in \mathcal{V}_+$, $Uv \in \mathcal{V}_-$, and $\mathcal{V}_+ \perp \mathcal{V}_-$.

2. Let \mathcal{S} be a J -self-orthogonal subspace. If

$$v \in \mathcal{S}, v = v_+ + v_-, v_{\pm} \in \mathcal{V}_{\pm},$$

then the condition $v \perp_J v = 0$, that is the condition $vJv^* = 0$ means that $v_1v_1^* = v_2v_2^*$. Therefore, if $v_1 = 0$, then also $v = 0$. This means that the projection mapping $v \rightarrow vP_+$, considered as a mapping from $\mathcal{S} \rightarrow \mathcal{V}_+$, is injective. For J -self-orthogonal subspace \mathcal{S} of the space \mathcal{V} , the equality $\dim \mathcal{S} = \dim \mathcal{V} - \dim \mathcal{S}$ holds. Hence $\dim \mathcal{S} = \dim \mathcal{V}_+$. Therefore, the injective linear mapping $v \rightarrow P_+$ is surjective. The inverse mapping is defined on the whole subspace \mathcal{V}_+ and can be presented in the form $v = v_1 + v_1U$, where U is a linear operator acting from \mathcal{V}_+ into \mathcal{V}_- . This mapping $v_1 \rightarrow v_1 + v_1U$ maps the subspace \mathcal{V}_+ onto the subspace \mathcal{S} .

Since $vJv^* = 0$, then $v_1v_1^* = v_2v_2^*$, where $v_2 = v_1U$. Since $v_1 \in \mathcal{V}_+$ is arbitrary, this means that the operator U is isometric. Since $\dim \mathcal{V}_+ = \dim \mathcal{V}_-$, the operator U is unitary. Thus, the originally given J -self-complementary subspace \mathcal{S} is of the form \mathcal{S}_U , where U is an unitary operator acting from \mathcal{V}_+ to \mathcal{V}_- .

The coincidence $\mathcal{S}_{U_1} = \mathcal{S}_{U_2}$ means that every vector of the form $v_1 + v_1U_1$, where $v_1 \in \mathcal{V}_+$ can also be presented in the form $v_2 + v_2U_2$ with some $v_2 \in \mathcal{V}_+$:

$$v_1 + v_1U_1 = v_2 + v_2U_2.$$

Since $v_1, v_2 \in \mathcal{V}_+$, $v_1U_1, v_2U_2 \in \mathcal{V}_-$, then $v_1 = v_2$, and $v_1U_1 = v_1U_2$. The equality $v_1U_1 = v_1U_2$ for every $v_1 \in \mathcal{V}_+$ means that $U_1 = U_2$. Thus, $(\mathcal{S}_{U_1} = \mathcal{S}_{U_2}) \Rightarrow (U_1 = U_2)$. \square

Choosing the orthogonal bases (4.45) in the subspaces \mathcal{V}_+ and \mathcal{V}_- , we represent an unitary operator U by the appropriate unitary matrix:

$$\begin{aligned} e_+^1 U &= e_-^1 u_{11} + e_-^2 u_{21}, \\ e_+^2 U &= e_-^1 u_{12} + e_-^2 u_{22}. \end{aligned}$$

The following result is a reformulation of Lemma 4.7:

Lemma 4.8. *Let \mathcal{V} be the space \mathbb{C}^4 of four vector-rows, J be a matrix of the form (4.37). With every 2×2 matrix $U = \|u_{pq}\|_{1 \leq p, q \leq 2}$, we associate the pair of vectors $v^1(U), v^2(U)$:*

$$v^1(U) = e_+^1 + e_-^1 u_{11} + e_-^2 u_{21}, \quad (4.47a)$$

$$v^2(U) = e_+^2 + e_-^1 u_{12} + e_-^2 u_{22}, \quad (4.47b)$$

where e_{\pm}^k , $k = 1, 2$, are the vector-rows of the form (4.45), and the subspace \mathcal{S}_U of \mathcal{V} which is the linear hull of the vectors $v^1(U)$, $v^2(U)$,

$$\mathcal{S}_U = \text{hull}(v^1(U), v^2(U)).$$

1. If the matrix U is unitary, then the vectors $v^1(U)$, $v^2(U)$ are linearly independent, and the subspace \mathcal{S}_U is J -self-complementary.
2. Let \mathcal{S} be a J -self-complementary subspace of the space \mathcal{V} . Then $\mathcal{S} = \mathcal{S}_U$ for some unitary matrix U .
3. For unitary matrices U_1, U_2 ,

$$(\mathcal{S}_{U_1} = \mathcal{S}_{U_2}) \Leftrightarrow (U_1 = U_2).$$

The "coordinate" form of the vectors $v^1(U)$, $v^2(U)$ is:

$$\begin{aligned} v^1(U) &= [1 + u_{11}, i(1 - u_{11}), u_{21}, -iu_{21}], \\ v^2(U) &= [u_{12}, -iu_{12}, 1 + u_{22}, i(1 - u_{22})]. \end{aligned} \quad (4.48)$$

Remembering, see Lemma 4.6, how J -self-complementary subspaces of the space \mathcal{V} are related to Ω_L -self-complementary subspaces the space $\mathcal{E}_L = \mathcal{D}_{\mathcal{L}_{\max}}/\mathcal{D}_{\mathcal{L}_{\min}}$ we formulate the following result

Lemma 4.9. *Let us associate with every 2×2 matrix $U = \|u_{pq}\|_{1 \leq p, q \leq 2}$ the pair of vectors $d^1(U)$, $d^2(U)$ of the space \mathcal{E}_L :*

$$d^1(U) = (1 + u_{11})\varphi_- + i(1 - u_{11})\psi_- + u_{21}\varphi_+ - iu_{21}\psi_+, \quad (4.49a)$$

$$d^2(U) = u_{12}\varphi_- - iu_{12}\psi_- + (1 + u_{22})\varphi_+ + i(1 - u_{22})\psi_+, \quad (4.49b)$$

where the functions φ_{\pm} , ψ_{\pm} are defined in (4.34). The subspace \mathcal{G}_U of the space \mathcal{E}_L is defined as the linear hull of the vectors $d^1(U)$, $d^2(U)$:

$$\mathcal{G}_U = \text{hull}(d^1(U), d^2(U)). \quad (4.50)$$

1. If the matrix U is unitary, then the subspace $\mathcal{S} = \mathcal{G}_U$ is Ω_L -self-complementary.
2. Let \mathcal{S} be a Ω_L -self-complementary subspace of the space \mathcal{E}_L . Then $\mathcal{S} = \mathcal{G}_U$ for some unitary matrix U .
3. For unitary matrices U_1, U_2 ,

$$(\mathcal{G}_{U_1} = \mathcal{G}_{U_2}) \Leftrightarrow (U_1 = U_2).$$

It is clear that a subspace $\mathcal{S} \subseteq \mathcal{E}_L$ is an Ω_L -self-complementary subspace if and only if its Ω_L -orthogonal complement $\mathcal{S}^{\perp_{\Omega_L}}$ is an Ω_L -self-complementary subspace. The subspace $(\mathcal{S}_U)^{\perp_{\Omega_L}}$ can be described as:

$$(\mathcal{S}_U)^{\perp_{\Omega_L}} = \{x \in \mathcal{E}_L : \Omega_L(x, d^1(U)) = 0, \Omega_L(x, d^2(U)) = 0\},$$

where d^1, d^2 are defined in (4.49), (4.34). Thus Lemma 4.9 can be reformulated in the following way:

Lemma 4.10. *Let us associate the pair of vectors $d^1(U), d^2(U)$ with every 2×2 matrix $U = \|u_{pq}\|_{1 \leq p, q \leq 2}$ by (4.49), (4.34). The subspace \mathcal{O}_U is defined as*

$$\mathcal{O}_U = \{x \in \mathcal{E}_L : \Omega_L(x, d^1(U)) = 0, \Omega_L(x, d^2(U)) = 0\}. \quad (4.51)$$

1. *If the matrix U is unitary, then the subspace $\mathcal{S} = \mathcal{O}_U$ is Ω_L -self-complementary.*
2. *Let \mathcal{S} be a Ω_L -self-complementary subspace of the space \mathcal{E}_L . Then $\mathcal{S} = \mathcal{O}_U$ for some unitary matrix U .*
3. *For unitary matrices U_1, U_2 ,*

$$(\mathcal{O}_{U_1} = \mathcal{O}_{U_2}) \Leftrightarrow (U_1 = U_2).$$

Thus there is one-to-one correspondence between the set of all 2×2 unitary matrices $U = \|u_{pq}\|_{1 \leq p, q \leq 2}$ and the set of all Ω_L -self-complementary subspaces \mathcal{S} of the space $\mathcal{E}_L = \mathcal{D}_{\mathcal{L}_{\max}}/\mathcal{D}_{\mathcal{L}_{\min}}$. This correspondence is described as

$$\mathcal{S} = \mathcal{O}_U, \quad (4.52)$$

where \mathcal{O}_U is defined in (4.51), (4.49), (4.34).

On the other hand, the subspaces of the space $\mathcal{E}_L = \mathcal{D}_{\mathcal{L}_{\max}}/\mathcal{D}_{\mathcal{L}_{\min}}$ which are self-complementary with respect to the Hermitian form Ω_L , (4.5), are in one-to-one correspondence to self-adjoint differential operators generated by the formal differential operator L , (4.11). Every self-adjoint differential operators \mathcal{L} generated by the formal differential operator L is the *restriction* of the maximal differential operator \mathcal{L}_{\max} , (4.3), on the appropriate domain of definition. According to Lemma 4.4, as applied to the operators $A_0 = \mathcal{L}_{\min}$, $A_0^* = \mathcal{L}_{\max}$, the domains of definition of a selfadjoint extension \mathcal{L} of the operator \mathcal{L}_{\min} are those subspaces \mathcal{S} :

$$\mathcal{D}_{\mathcal{L}_{\min}} \subseteq \mathcal{S} \subseteq \mathcal{D}_{\mathcal{L}_{\max}} \quad (4.53)$$

which are self-complementary with respect to the Hermitian form Ω_L , (4.5). According to Lemma 4.10, Ω_L -self-complementary subspaces \mathcal{S} can be described by means of the conditions

$$\mathcal{S} = \{x(t) \in \mathcal{D}_{\mathcal{L}_{\max}} : \Omega_L(x, d^1(U)) = 0, \Omega_L(x, d^2(U)) = 0\}, \quad (4.54)$$

where $d^1(U)$, $d^2(U)$ are the same that in (4.49), (4.34), U is an unitary 2×2 matrix.

The conditions $\Omega_L(x, d^1(U)) = 0$, $\Omega_L(x, d^2(U)) = 0$ may be interpreted as a *boundary conditions* posed on functions $x \in \mathcal{D}_{\mathcal{L}_{\max}}$. Let us present these conditions in more traditional form.

Let us introduce the following notations:

$$\begin{aligned} b_{-a}(x) &= \lim_{t \rightarrow -a+0} (t+a) \frac{dx(t)}{dt}, & b_a(x) &= \lim_{t \rightarrow a-0} (t-a) \frac{dx(t)}{dt}, \\ c_{-a}(x) &= \lim_{t \rightarrow -a+0} \left((t+a) \ln(a+t) \frac{dx(t)}{dt} - x(t) \right), \\ c_a(x) &= \lim_{t \rightarrow a-0} \left((t-a) \ln(a-t) \frac{dx(t)}{dt} - x(t) \right). \end{aligned} \quad (4.55)$$

Remark 4.1. The values $b_{-a}(x)$, $c_{-a}(x)$ and $b_a(x)$, $c_a(x)$ may be considered as generalized boundary values related to the function $x(t) \in \mathcal{D}_{\mathcal{L}_{\max}}$ at the endpoints $-a$ and a of the interval $(-a, a)$.

Remark 4.2. The solutions x_1^- , x_2^- , (x_1^+, x_2^+) of the equation $Lx = \lambda x$, which appears in Lemma 4.1, satisfy the conditions

$$\begin{aligned} b_{-a}(x_1^-) &= 0, & c_{-a}(x_1^-) &= -1; & b_{-a}(x_2^-) &= 1, & c_{-a}(x_2^-) &= 0, \\ b_a(x_1^+) &= 0, & c_a(x_1^+) &= -1; & b_a(x_2^+) &= 1, & c_a(x_2^+) &= 0. \end{aligned}$$

Lemma 4.11. For $x(t) \in \mathcal{D}_{\mathcal{L}_{\max}}$, the limits (4.55) exist, are finite, and

$$b_{-a}(x) = \frac{ia}{2} \Omega_L(x, \varphi_-), \quad c_{-a}(x) = \frac{ia}{2} \Omega_L(x, \psi_-), \quad (4.56a)$$

$$b_a(x) = \frac{ia}{2} \Omega_L(x, \varphi_+), \quad c_a(x) = \frac{ia}{2} \Omega_L(x, \psi_+), \quad (4.56b)$$

where the functions $\varphi_{\pm}, \psi_{\pm}$ are defined in (4.34), and the form Ω_L is defined by (4.5).

Proof. The existence of the limits in (4.56) follows from Lemma 4.5 applied to the functions $x(t)$ and $y(t) = \varphi_{\pm}(t)$ or $y(t) = \psi_{\pm}(t)$. The equalities (4.56) can be obtained by the direct computation using the explicit expressions (4.34) for the functions $\varphi_{\pm}(t)$, $\psi_{\pm}(t)$. \square

Due to (4.56), the equality (4.36) can be rewritten as

$$\begin{bmatrix} b_{-a}(\varphi_-) & c_{-a}(\varphi_-) & b_{-a}(\varphi_-) & c_a(\varphi_-) \\ b_{-a}(\psi_-) & c_{-a}(\psi_-) & b_{-a}(\psi_-) & c_a(\psi_-) \\ b_{-a}(\varphi_+) & c_{-a}(\varphi_+) & b_{-a}(\varphi_+) & c_a(\varphi_+) \\ b_{-a}(\psi_+) & c_{-a}(\psi_+) & b_{-a}(\psi_+) & c_a(\psi_+) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (4.57)$$

Remark 4.3. *The characterization (4.38) of the domain of definition $\mathcal{D}_{\mathcal{L}_{\min}}$ of the minimal operator \mathcal{L}_{\min} can be presented as:*

$$\mathcal{D}_{\mathcal{L}_{\min}} = \{x(t) : x(t) \in \mathcal{D}_{\mathcal{L}_{\max}}, \text{ and } b_{-a}(x) = 0, c_{-a}(x) = 0, b_a(x) = 0, c_a(x) = 0\}, \quad (4.58)$$

In view of (4.56), the equalities $\Omega_L(x, d^1(U) = 0, \Omega_L(x, d^2(U) = 0)$ take the form

$$(1 + u_{11}) b_{-a}(x) - i(1 - u_{11}) c_{-a}(x) + u_{12} b_a(x) + iu_{12} c_a(x) = 0, \quad (4.59a)$$

$$u_{21} b_{-a}(x) + iu_{21} c_{-a}(x) + (1 + u_{22}) b_a(x) - i(1 - u_{22}) c_a(x) = 0 \quad (4.59b)$$

Remark 4.4. *Since the form $\Omega_L(x, y)$ is antilinear with respect to the argument y : $\Omega_L(x, \mu y) = \overline{\mu} \Omega_L(x, y)$ for $\mu \in \mathbb{C}$, the numbers $i, -i$ which occurs in (4.49) must be replaced with the numbers $-i, i$ in appropriate positions in the equality (4.59). For the same reason, the numbers u_{pq} which occurs in (4.49) must be replaced with the numbers $\overline{u_{pq}}$ in (4.59). However to simplify the notation, we replace the number u_{pq} with the number u_{qp} rather with the numbers $\overline{u_{pq}}$. This corresponds to that in (4.49) we use the matrix U^* rather than U as a matrix which parameterizes the set of all Ω_L -self-orthogonal subspaces. The matrix U^* is an arbitrary unitary matrix as well the matrix U .*

Definition 4.9. *Let U be a 2×2 matrix. The operator \mathcal{L}_U is defined in the following way:*

1. *The domain of definition $\mathcal{D}_{\mathcal{L}_U}$ of the operator \mathcal{L}_U is the set of all $x(t) \in \mathcal{D}_{\mathcal{L}_{\max}}$ which satisfy the conditions (4.59a)-(4.59b), (4.55).*
2. *For $x \in \mathcal{D}_{\mathcal{L}_U}$, the action of the operator \mathcal{L}_U is: $\mathcal{L}_U x = \mathcal{L}_{\max} x$.*

Remark 4.5. *In view of (4.59) and (4.59), for any matrix U ,*

$$\mathcal{D}_{\mathcal{L}_{\min}} \subseteq \mathcal{D}_{\mathcal{L}_U}.$$

Thus for any matrix U , the operator \mathcal{L}_U is an extension of the operator \mathcal{L}_{\min} :

$$\mathcal{L}_{\min} \subseteq \mathcal{L}_U \subseteq \mathcal{L}_{\max}. \quad (4.60)$$

The equalities (4.59a) which determine the domain of definition of the extension \mathcal{L}_U can be considered as boundary conditions posed on functions $x \in \mathcal{D}_{\mathcal{L}_{\max}}$. (See Remark 4.1.)

The following Theorem is a reformulation of Lemma 4.10 in the language of extensions of operators.

Theorem 4.1.

1. If U is an unitary matrix, then the operator \mathcal{L}_U is a selfadjoint differential operator which is an extension of the minimal differential operator \mathcal{L}_{\min} : $\mathcal{L}_{\min} \subset \mathcal{L}_U \subset \mathcal{L}_{\max}$.
2. Every differential operator \mathcal{L} which is a selfadjoint extension of the minimal differential operator \mathcal{L}_{\min} , is of the form $\mathcal{L} = \mathcal{L}_U$ for some unitary matrix U .
3. For unitary matrices U_1, U_2 ,

$$(U_1 = U_2) \Leftrightarrow (\mathcal{L}_{U_1} = \mathcal{L}_{U_2}).$$

Commutator of the operator $\mathcal{F}_{[-a,a]}$ and \mathcal{L}_U .

Let us calculate the difference $\mathcal{F}_{[-a,a]}\mathcal{L}_{\max}x - \mathcal{L}_{\max}\mathcal{F}_{[-a,a]}x$ for $x \in \mathcal{D}_{\mathcal{L}_{\max}}$. Notice that $\mathcal{L}_{\max}x \in L^2([-a, a])$, so $\mathcal{F}_{[-a,a]}(\mathcal{L}_{\max}x)$ is defined. Since $x \in L^2([-a, a])$, the function $\mathcal{F}_{[-a,a]}x(t)$ is smooth on the closed interval $[-a, a]$. (In fact this function is analytic in the whole real axis.) All the more, $\mathcal{F}_{[-a,a]}x \in \mathcal{D}_{\mathcal{L}_{\max}}$. Thus for $x \in \mathcal{D}_{\mathcal{L}_{\max}}$, the difference $\mathcal{F}_{[-a,a]}\mathcal{L}_{\max}x - \mathcal{L}_{\max}\mathcal{F}_{[-a,a]}x$ is well defined.

Assuming that $x \in \mathcal{D}_{\mathcal{L}_{\max}}$ and that $-a < \alpha < \beta < a$, we integrate

by parts twice⁷ :

$$\begin{aligned}
& \int_{\alpha}^{\beta} \left(-\frac{d}{d\xi} \left(\left(1 - \frac{\xi^2}{a^2} \right) \right) \frac{dx(\xi)}{d\xi} \right) e^{it\xi} d\xi = \\
& = - \left(1 - \frac{\xi^2}{a^2} \right) \frac{dx(\xi)}{d\xi} e^{it\xi} \Big|_{\xi=\alpha}^{\xi=\beta} + it \left(1 - \frac{\xi^2}{a^2} \right) x(\xi) e^{it\xi} \Big|_{\xi=\alpha}^{\xi=\beta} - \\
& \quad - it \int_{\alpha}^{\beta} x(\xi) \frac{d}{d\xi} \left(\left(1 - \frac{\xi^2}{a^2} \right) e^{it\xi} \right) d\xi. \quad (4.61)
\end{aligned}$$

For $x \in \mathcal{D}_{\mathcal{L}_{\max}}$, both limits $\lim_{t \rightarrow \pm a} (1 - t^2/a^2) \frac{dx(t)}{dt}$ exist, are finite, and

$$\lim_{t \rightarrow -a} (1 - t^2/a^2) \frac{dx(t)}{dt} = \frac{2}{a^2} b_{-a}(x), \quad (4.62a)$$

$$\lim_{t \rightarrow +a} (1 - t^2/a^2) \frac{dx(t)}{dt} = -\frac{2}{a^2} b_a(x). \quad (4.62b)$$

where $b_{-a}(x), b_a(x)$ are defined in (4.55) and also appear in the boundary conditions (4.59). Since the limits in (4.62) are finite, we conclude that $|x(t)| = O(\ln(a^2 - t^2))$ as $t \rightarrow \pm a, |t| < a$. All the more, for $x \in \mathcal{D}_{\mathcal{L}_{\max}}$

$$\lim_{t \rightarrow -a+0} \left(1 - \frac{t^2}{a^2} \right) x(t) = 0. \quad (4.63)$$

Passing to the limit in (4.61) and taking into account (4.63) and (4.62), we obtain

$$\begin{aligned}
& \int_{-a}^a \left(-\frac{d}{d\xi} \left(\left(1 - \frac{\xi^2}{a^2} \right) \frac{dx(\xi)}{d\xi} \right) \right) e^{it\xi} d\xi = \frac{2}{a} \left(b_+(x) e^{iat} + b_-(x) e^{-iat} \right) - \\
& \quad - it \int_{-a}^a x(\xi) \frac{d}{d\xi} \left(\left(1 - \frac{\xi^2}{a^2} \right) e^{it\xi} \right) d\xi. \quad (4.64)
\end{aligned}$$

Transforming the last summand of the right hand side of (4.64), we

⁷ Like it is done in (2.31) of the manuscript [KaMa].

obtain

$$\begin{aligned}
& -it \int_{-a}^a x(\xi) \frac{d}{d\xi} \left(\left(1 - \frac{\xi^2}{a^2} \right) e^{it\xi} \right) d\xi = \\
& = t^2 \int_{-a}^a x(\xi) e^{it\xi} d\xi + \frac{it}{a^2} \int_{-a}^a x(\xi) \frac{d}{d\xi} (\xi^2 e^{it\xi}) d\xi = \\
& \left(\text{since } \frac{d}{d\xi} (\xi^2 e^{it\xi}) = \frac{d}{d\xi} \left(-\frac{d^2}{dt^2} e^{it\xi} \right) = -\frac{d^2}{dt^2} (ite^{it\xi}) \right) \\
& = t^2 \int_{-a}^a x(\xi) e^{it\xi} d\xi + \frac{t}{a^2} \frac{d^2}{dt^2} \left(t \int_{-a}^a x(\xi) e^{it\xi} d\xi \right) = \\
& = t^2 \int_{-a}^a x(\xi) e^{it\xi} d\xi + \frac{d}{dt} \left(\frac{t^2}{a^2} \frac{d}{dt} \int_{-a}^a x(\xi) e^{it\xi} d\xi \right) = \\
& = t^2 \int_{-a}^a x(\xi) e^{it\xi} d\xi - \frac{d}{dt} \left(\left(1 - \frac{t^2}{a^2} \right) \frac{d}{dt} \int_{-a}^a x(\xi) e^{it\xi} d\xi \right) - \int_{-a}^a \xi^2 x(\xi) e^{it\xi} d\xi.
\end{aligned} \tag{4.65}$$

Unifying (4.64) and (4.65), we obtain the equality

$$\begin{aligned}
& \int_{-a}^a \left(\left(-\frac{d}{d\xi} \left(1 - \frac{\xi^2}{a^2} \right) \frac{d}{d\xi} + \xi^2 \right) x(\xi) \right) e^{it\xi} d\xi = \\
& = \frac{2}{a} \left(b_+(x) e^{iat} + b_-(x) e^{-iat} \right) + \left(-\frac{d}{dt} \left(1 - \frac{t^2}{a^2} \right) \frac{d}{dt} + t^2 \right) \int_{-a}^a x(\xi) e^{it\xi} d\xi.
\end{aligned} \tag{4.66}$$

We summarize the above calculation as

Lemma 4.12. *Let $\mathcal{F}_{[-a,a]}$ be the Fourier operator truncated on the finite symmetric interval $[-a, a]$. Let \mathcal{L}_{\max} be the maximal differential operator with domain of definition $\mathcal{D}_{\mathcal{L}_{\max}}$ generated by the formal differential operator $L = -\frac{d}{dt} \left(1 - \frac{t^2}{a^2} \right) \frac{d}{dt} + t^2$. (See Definition 4.3.)*

If $x \in \mathcal{D}_{\mathcal{L}_{\max}}$, then $\mathcal{F}_{[-a,a]}x \in \mathcal{D}_{\mathcal{L}_{\max}}$, and the equality holds

$$(\mathcal{F}_{[-a,a]} \mathcal{L}_{\max} x)(t) - (\mathcal{L}_{\max} \mathcal{F}_{[-a,a]} x)(t) = \frac{2}{a} \left(b_+(x) e^{iat} + b_-(x) e^{-iat} \right). \tag{4.67}$$

Every selfadjoint differential operator generated by the formal differential operator L is a restriction of the maximal differential operator \mathcal{L}_{\max} on the appropriate domain of definition. According to Theorem 4.1, the set of such self-adjoint operators coincides with the set of operators \mathcal{L}_U , where U is an arbitrary 2×2 unitary matrix. The domain of definition $\mathcal{D}_{\mathcal{L}_U}$ of the operator \mathcal{L}_U is distinguished from the domain $\mathcal{D}_{\mathcal{L}_{\max}}$ by the boundary conditions (4.59) constructed from U . The next theorem answers the question which operators \mathcal{L}_U commute with the truncated Fourier operator $\mathcal{F}_{[-a,a]}$.

Theorem 4.2.

1. If $U = I$, where I is 2×2 identity matrix, then the differential operator⁸ \mathcal{L}_I commutes with the truncated⁹ Fourier operator $\mathcal{F}_{[-a,a]}$:

$$\mathcal{F}_{[-a,a]}\mathcal{L}_I x = \mathcal{L}_I\mathcal{F}_{[-a,a]} x \quad \forall x \in \mathcal{D}_{\mathcal{L}_I}. \quad (4.68)$$

2. If $U \neq I$, then the operator \mathcal{L}_U do not commute with the operator $\mathcal{F}_{[-a,a]}$:

- a. There exist vectors $x \in \mathcal{D}_{\mathcal{L}_U}$ such that $\mathcal{F}_{[-a,a]}x \in \mathcal{D}_{\mathcal{L}_U}$, so both operators $\mathcal{F}_{[-a,a]}\mathcal{L}_U$ and $\mathcal{L}_U\mathcal{F}_{[-a,a]}$ are applicable to x , but

$$\mathcal{F}_{[-a,a]}\mathcal{L}_U x \neq \mathcal{L}_U\mathcal{F}_{[-a,a]}x; \quad (4.69)$$

- b. There exist vectors $x \in \mathcal{D}_{\mathcal{L}_U}$ such that $\mathcal{F}_{[-a,a]}x \notin \mathcal{D}_{\mathcal{L}_U}$, so the operator $\mathcal{L}_U\mathcal{F}_{[-a,a]}$ even can not be applied to such x .

Proof.

1. For $U = I$, the boundary conditions (4.59) take the form

$$b_{-a}(x) = 0, \quad b_a(x) = 0. \quad (4.70)$$

Thus, the domain of definition $\mathcal{D}_{\mathcal{L}_I}$ of the operator \mathcal{L}_I is:

$$\mathcal{D}_{\mathcal{L}_I} = \{x : x \in \mathcal{D}_{\mathcal{L}_{\max}}, b_{-a}(x) = 0, b_a(x) = 0\}. \quad (4.71)$$

Every smooth function $x(t)$ on $(-a, a)$ which derivative is bounded: $\sup_{t \in (-a, a)} |x'(t)| < \infty$, belongs to $\mathcal{D}_{\mathcal{L}_{\max}}$. Moreover, according to (4.55), every such a function satisfies the boundary condition (4.71), i.e. $b_{-a}(x) = 0, b_a(x) = 0$. Hence *every smooth function on $(-a, a)$ which derivative is bounded on $(-a, a)$, belongs to domain of definition*

⁸ $\mathcal{L}_I = \mathcal{L}_U$ for $U = I$.

⁹ $\mathcal{F}_{[-a,a]} = F_E$ for $E = [-a, a]$.

$\mathcal{D}_{\mathcal{L}_I}$ of the operator \mathcal{L}_I . In particular, if $x \in L^2((-a, a)$ and $y = \mathcal{F}_{[-a, a]}x$, then $y \in \mathcal{D}_{\mathcal{L}_I}$. Thus for $x \in \mathcal{D}_{\mathcal{L}_I}$ both summands in the expression $\mathcal{F}_{[-a, a]}\mathcal{L}_Ix - \mathcal{L}_I\mathcal{F}_{[-a, a]}x$ are well defined. Since the operator \mathcal{L}_I is a restriction of the operator \mathcal{L}_{\max} , then

$$\mathcal{F}_{[-a, a]}\mathcal{L}_Ix - \mathcal{L}_I\mathcal{F}_{[-a, a]}x = \mathcal{F}_{[-a, a]}\mathcal{L}_{\max}x - \mathcal{L}_{\max}\mathcal{F}_{[-a, a]}x \quad \text{for } x \in \mathcal{D}_{\mathcal{L}_I}.$$

In view of (4.67) and (4.70), the equality (4.68) holds.

2. Let $U \neq I$. Then at least of one value $u_{11} - 1$ or $u_{22} - 1$ differs from zero. For definiteness, let $u_{11} - 1 \neq 0$. Set

$$\gamma = \frac{1 + u_{11}}{i(1 - u_{11})}, \quad x(t) = \psi_-(t) + \gamma\varphi_-(t) + x_0(t), \quad (4.72)$$

where $x_0(t)$ is a smooth function which support is a compact subset of the *open* interval $(-a, a)$:

$$\text{supp } x_0 \subseteq (-a, a). \quad (4.73)$$

The function x_0 will be chosen later. According to (4.57), (4.73) and the choice of γ , *for any choice of* $x_0(t)$, the function $x(t)$ from (4.72) satisfy the boundary conditions (4.59). Thus,

$$x(t) \in \mathcal{D}_{\mathcal{L}_U}. \quad (4.74)$$

for any choice of x_0 . Moreover

$$b_{-a}(x) = 1, \quad b_a(x) = 0. \quad (4.75)$$

For the function $y(t) = (\mathcal{F}_{(-a, a)}x)(t)$, *the boundary conditions* (4.59) *either hold, or does not hold*. This depends on the choice of the function x_0 . If (4.59) hold for this y , then $\mathcal{F}_{(-a, a)}x \in \mathcal{D}_{\mathcal{L}_U}$ and the equality (4.67) can be interpreted as the equality

$$(\mathcal{F}_{(-a, a)}\mathcal{L}_Ux)(t) - (\mathcal{L}_U\mathcal{F}_{(-a, a)}x)(t) = \frac{2}{a} \left(b_+(x)e^{iat} + b_-(x)e^{-iat} \right). \quad (4.76)$$

In view of (4.75), $(\mathcal{F}_{(-a, a)}\mathcal{L}_Ux)(t) - (\mathcal{L}_U\mathcal{F}_{(-a, a)}x)(t) \neq 0$.

Let us show that both of the possibilities $\mathcal{F}_{(-a, a)}x \in \mathcal{D}_{\mathcal{L}_U}$ and $\mathcal{F}_{(-a, a)}x \notin \mathcal{D}_{\mathcal{L}_U}$ are realizable. Since the function $y(t) = (\mathcal{F}_{(-a, a)}x)(t)$ is smooth on $[-a, a]$,

$$b_{-a}(y) = 0, \quad b_a(y) = 0, \quad c_{-a}(y) = -y(-a), \quad c_a(y) = -y(a).$$

Thus as applied to the function y , the boundary conditions (4.59) take the form

$$(1 - u_{11})y(-a) - u_{12}y(a) = 0, \quad (4.77a)$$

$$u_{21}y(-a) - (1 - u_{22})y(a) = 0. \quad (4.77b)$$

If, using the freedom of choice of the function $x_0(t)$ in (4.72), we can arbitrary prescribe the values $y(-a)$ and $y(a)$, then we can either satisfy the boundary conditions (4.77) (prescribing $y(-a) = 0$, $y(a) = 0$), or violate them (if $u_{11} \neq 1$, we prescribe $y(-a) = 1$, $y(a) = 0$, if $u_{11} \neq 1$, we prescribe $y(-a) = 0$, $y(a) = 1$.) The reference to Lemma below finishes the proof. \square

Lemma 4.13. *Given complex numbers y_1 and y_2 , there exists a smooth function $x_0(t)$ on $[-a, a]$ which possesses the properties:*

1. $\text{supp } x_0 \subseteq (-a, a)$.
2. $y_0(-a) = y_1$, $y_0(a) = y_2$, where $y_0 = \mathcal{F}_{[-a, a]}(x_0)$.

Proof. The evaluations $y(-a)$ and $y(a)$ are linearly independent linear functionals on the space of functions on $(-a, a)$ which are smooth and compactly supported:

$$y(-a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^a x(\xi) e^{-ia\xi} d\xi, \quad y(a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^a x(\xi) e^{ia\xi} d\xi,$$

and the functions $e^{-ia\xi}$, $e^{ia\xi}$ generating these linear functionals are linearly independent on any non-empty open subinterval of the interval $(-a, a)$. \square

Properties of the operator \mathcal{L}_I . As we have established, Theorems 4.1 and 4.2, the only selfadjoint differential operator which is generated by the formal operator L and which commutes with the truncated Fourier operator $\mathcal{F}_{[-a, a]}$ is the operator \mathcal{L}_I . Let us discuss properties of the operator \mathcal{L}_I .

The following lemma gives an alternative definition of the domain $\mathcal{D}_{\mathcal{L}_I}$.

Lemma 4.14. *Let a function $x(t)$ belong to $\mathcal{D}_{\mathcal{L}_{\max}}$. Then the function $x(t)$ belong to $\mathcal{D}_{\mathcal{L}_I}$ if and only if both limits*

$$x(-a) = \lim_{t \rightarrow -a+0} x(t), \quad x(a) = \lim_{t \rightarrow a-0} x(t). \quad (4.78)$$

exist and are finite.

Proof. 1. Functions $x(t)$ belonging to $\mathcal{D}_{\mathcal{L}_I}$ possess the properties

$$\lim_{t \rightarrow \pm a} (a^2 - t^2) \frac{dx(t)}{dt} = 0, \quad (4.79)$$

$$\int_{-a}^a \left| \frac{d}{d\xi} \left((a^2 - \xi^2) \frac{dx(\xi)}{d\xi} \right) \right|^2 d\xi = C^2 < \infty, \quad C > 0. \quad (4.80)$$

From (4.80) and the Schwarz inequality it follows that

$$\int_{t_1}^{t_2} \left| \frac{d}{d\xi} \left((a^2 - \xi^2) \frac{dx(\xi)}{d\xi} \right) \right| d\xi \leq C \sqrt{t_2 - t_1}, \quad -a < t_1 < t_2 < a.$$

All the more,

$$\left| (a^2 - \xi^2) \frac{dx(\xi)}{d\xi} \right|_{\xi=t_1}^{\xi=t_2} \leq C \sqrt{t_2 - t_1}.$$

We use the last inequality for $t_1 = -a + 0$, $t_2 = t$, where $-a < t < a$. Taking into account (4.79), we deduce that

$$\left| (a^2 - t^2) \frac{dx(t)}{dt} \right| \leq C \sqrt{a + t}, \quad -a \leq t \leq 0.$$

Analogously,

$$\left| (a^2 - t^2) \frac{dx(t)}{dt} \right| \leq C \sqrt{a - t}, \quad 0 \leq t \leq a.$$

From two last inequalities it follows that

$$\left| (a^2 - t^2) \frac{dx(t)}{dt} \right| \leq \frac{1}{\sqrt{a}} \sqrt{a^2 - t^2}, \quad -a < t < a.$$

Finally, from (4.79) and (4.80) we deduced the inequalities

$$\left| \frac{dx(t)}{dt} \right| \leq \frac{C}{\sqrt{a}} \frac{1}{\sqrt{a^2 - t^2}}, \quad -a < t < a. \quad (4.81)$$

and

$$|x(t_2) - x(t_1)| \leq \frac{C}{\sqrt{a}} \int_{t_1}^{t_2} \frac{d\xi}{\sqrt{a^2 - \xi^2}}, \quad -a < t_1 < t_2 < a. \quad (4.82)$$

Since $\int_{-a}^a \frac{dt}{\sqrt{a^2-t^2}} < \infty$, the function $x(t)$ is uniformly continuous on the interval $(-a, a)$. Therefore the limits (4.78) exist and are finite.

2. According to Lemma 4.11, both limits $\lim_{t \rightarrow \pm(a-0)} (t \pm a) \frac{dx(t)}{dt}$ exist and are finite. If $x \notin \mathcal{D}_{\mathcal{L}_I}$, then at least one of these limits is not zero. If for example $\lim_{t \rightarrow a-0} (t-a) \frac{dx(t)}{dt} \neq 0$, then the function $x(t)$ grows logarithmically as $t \rightarrow a-0$. \square

Theorem 4.3.

1. *The selfadjoint operator \mathcal{L}_I is an operator with discrete spectrum.*
2. *The spectrum of the operator \mathcal{L}_I is a sequence of positive eigenvalues of multiplicity one which tends to $+\infty$.*
3. *The number λ is an eigenvalue of the differential operator \mathcal{L}_I if and only if there exists the non-zero solution $e(t, \lambda)$ of the boundary value problem for the differential equation*

$$-\frac{d}{dt} \left(\left(1 - \frac{t^2}{a^2} \right) \frac{de(t, \lambda)}{dt} \right) + t^2 e(t, \lambda) = \lambda e(t, \lambda) \quad (4.83a)$$

with the boundary conditions

$$e(-a, \lambda) \text{ is finite, } e(a, \lambda) \text{ is finite.} \quad (4.83b)$$

This solution $e(t, \lambda)$ is an eigenvector of the operator \mathcal{L}_I corresponding to the eigenvalue λ .

Remark 4.6. *The solutions of the boundary value problem (4.83) are known as the prolate spheroidal wave functions. There is a literature where these functions are discussed and studied. See for example [ChSt], [Fl], [KPS], [MSch], [SMCLC].*

If A is a symmetric operator in a Hilbert space \mathfrak{H} which domain of definition \mathcal{D}_A is dense in \mathfrak{H} and M is a bounded selfadjoint operator defined everywhere in \mathfrak{H} , then the operators A and $B = A + M$ ($\mathcal{D}_B = \mathcal{D}_A$) are selfadjoint or not simultaneously, and spectra of A and B are discrete or not simultaneously.

We use this fact in the case when $\mathfrak{H} = L^2((-a, a))$, $A = \mathcal{L}_I$, $Mx(t) = x(t) - t^2 x(t)$, so the operator B is a differential operator Λ of the form

$$(\Lambda x)(t) = -\frac{d}{dt} \left(\left(1 - \frac{t^2}{a^2} \right) \frac{dx(t)}{dt} \right) + x(t). \quad (4.84)$$

which domain of definition \mathcal{D}_Λ coincides with the domain of definition $\mathcal{D}_{\mathcal{L}_I}$ of the operator \mathcal{L}_I . (See (4.71) and (4.12a).)

Lemma 4.15. *Each of the operators \mathcal{L}_I and Λ is non-negative, and for every $x \in \mathcal{D}_{\mathcal{L}_I} = \mathcal{D}_\Lambda$ the equalities hold:*

$$\langle \mathcal{L}_I x, x \rangle = \int_{-a}^a \left(1 - \frac{\xi^2}{a^2}\right) \left| \frac{dx(\xi)}{d\xi} \right|^2 d\xi + \int_{-a}^a \xi^2 |x(\xi)|^2 d\xi, \quad (4.85)$$

$$\langle \Lambda x, x \rangle = \int_{-a}^a \left(1 - \frac{\xi^2}{a^2}\right) \left| \frac{dx(\xi)}{d\xi} \right|^2 d\xi + \int_{-a}^a |x(\xi)|^2 d\xi. \quad (4.86)$$

Proof. Let $-a < \alpha < \beta < a$. Integrating by parts we obtain

$$\begin{aligned} \int_{\alpha}^{\beta} \left(-\frac{d}{d\xi} \left(1 - \frac{\xi^2}{a^2}\right) \frac{dx(\xi)}{d\xi} \right) \overline{x(\xi)} d\xi = \\ = - \left(1 - \frac{\xi^2}{a^2}\right) \frac{dx(\xi)}{d\xi} \cdot \overline{x(\xi)} \Big|_{\xi=\alpha}^{\xi=\beta} + \int_{\alpha}^{\beta} \left(1 - \frac{\xi^2}{a^2}\right) \frac{dx(\xi)}{d\xi} \cdot \frac{\overline{dx(\xi)}}{d\xi} d\xi. \end{aligned}$$

According to the boundary conditions (4.70),

$$\lim_{t \rightarrow \pm(a-0)} \left(1 - \frac{t^2}{a^2}\right) \frac{dx(t)}{dt} = 0,$$

According to Lemma 4.14,

$$|x(t)| = O(1) \text{ as } |t| \rightarrow a - 0.$$

Passing to the limit as $\alpha \rightarrow -a + 0$, $\beta \rightarrow a - 0$, we obtain the equality

$$\int_{-a}^a \left(-\frac{d}{d\xi} \left(1 - \frac{\xi^2}{a^2}\right) \frac{dx(\xi)}{d\xi} \right) \overline{x(\xi)} d\xi = \int_{-a}^a \left(1 - \frac{\xi^2}{a^2}\right) \left| \frac{dx(\xi)}{d\xi} \right|^2 d\xi, \quad (4.87)$$

for every $x \in \mathcal{D}_{\mathcal{L}_I} = \mathcal{D}_\Lambda$.

□

Proof of Theorem 4.3. Let

$$\mathfrak{B} = \left\{ x \in \mathcal{D}_\Lambda : \langle \Lambda x, \Lambda x \rangle_{L^2(-a,a)} \leq 1 \right\}. \quad (4.88)$$

be a preimage of the unit ball of the space $L^2(-a, a)$ with respect to the mapping $x \rightarrow \Lambda x$. To prove that the spectrum of Λ is discrete it is enough to prove that the set \mathfrak{B} is precompact in $L^2(-a, a)$. The condition $\langle \Lambda x, \Lambda x \rangle_{L^2(-a, a)} \leq 1$ for a function $x \in \mathcal{D}_\Lambda$ means that

$$\int_{-a}^a \left| -\frac{d}{d\xi} \left(\left(1 - \frac{\xi^2}{a^2}\right) \frac{dx(\xi)}{d\xi} \right) + x(\xi) \right|^2 d\xi \leq 1 \quad (4.89)$$

In view of (4.87),

$$\begin{aligned} \int_{-a}^a \left| -\frac{d}{d\xi} \left(\left(1 - \frac{\xi^2}{a^2}\right) \frac{dx(\xi)}{d\xi} \right) \right|^2 d\xi + \int_{-a}^a |x(\xi)|^2 d\xi \leq \\ \int_{-a}^a \left| -\frac{d}{d\xi} \left(\left(1 - \frac{\xi^2}{a^2}\right) \frac{dx(\xi)}{d\xi} \right) + x(\xi) \right|^2 d\xi. \end{aligned}$$

Therefore from (4.89) it follows that

$$\int_{-a}^a |x(\xi)|^2 d\xi \leq 1 \quad (4.90)$$

and

$$\int_{-a}^a \left| -\frac{d}{d\xi} \left(\left(1 - \frac{\xi^2}{a^2}\right) \frac{dx(\xi)}{d\xi} \right) \right|^2 d\xi \leq 1. \quad (4.91)$$

Inequality (4.91) the inequality (4.80) for $C = a^2$. According to (4.82), the function x satisfy the inequality

$$|x(t_2) - x(t_1)| \leq a^{3/2} \int_{t_1}^{t_2} \frac{d\xi}{\sqrt{a^2 - \xi^2}}, \quad -a < t_1 < t_2 < a. \quad (4.92)$$

Thus, the set of the functions x belonging to \mathfrak{B} is uniformly bounded, (4.90), and equicontinuous, (4.92). Therefore, the set \mathfrak{B} is precompact in $L^2([-a, a])$.

Thus the spectra of the operators Λ and \mathcal{L}_I is discrete, i.e. consists of isolated eigenvalues. According to (4.85), the eigenvalues of the operator \mathcal{L}_I are positive. If λ is an eigenvalue of the operator \mathcal{L}_I and $e(t, \lambda)$ is an eigenfunction which corresponds to this λ , then, since $e(t, \lambda) \in \mathcal{D}_{\mathcal{L}_I}$, the function $e(t, \lambda)$ is continuous in t at the points $t = a$ and $t = -a$. (Lemma 4.14.)

Moreover the function $e(t, \lambda)$ is the solution of the differential equation $Lx = \lambda x$. As any solution of this equation, the function $e(t, \lambda)$ is a linear combination of the solutions $x_1^-(t, \lambda)$ and $x_2^-(t, \lambda)$. (The solutions $x_1^\pm(t, \lambda), x_2^\pm(t, \lambda)$ were introduced in Lemma 4.1). From the behavior of the functions $e(t, \lambda), x_1^-(t, \lambda), x_2^-(t, \lambda)$ by $t \rightarrow -a + 0$ we deduce that the function $e(t, \lambda)$ is proportional to $x_1^-(t, \lambda)$:

$$e(t, \lambda) = C_- x_1^-(t, \lambda), \quad C_- \neq 0 \text{ is a constant.}$$

Analogously,

$$e(t, \lambda) = C_+ x_1^+(t, \lambda), \quad C_+ \neq 0 \text{ is a constant.}$$

Thus, up to the proportionality, there is only one eigenfunction corresponding to the eigenvalue λ . \square

Remark 4.7. *Thus if λ is an eigenvalue of \mathcal{L}_I , then $C_- x_1^-(t, \lambda) = C_+ x_1^+(t, \lambda)$. Since the differential equation $Lx = \lambda x$ is invariant with respect to the change of variable $t \rightarrow -t$, then the functions $e(-t, \lambda), e(t, \lambda) \pm e(-t, \lambda)$ are eigenfunctions as well. Since there is only one eigenfunction up to proportionality, then either $e(t, \lambda) = e(-t, \lambda)$, or $e(t, \lambda) = -e(-t, \lambda)$. Thus, either $C_+ = C_-$, or $C_+ = -C_-$.*

Remark 4.8. *The spectral analysis of the operator Λ can be done explicitly. Its eigenfunctions are essentially the Legendre polynomials, the spectrum also can be found explicitly. The property of the spectrum of Λ to be discrete may be derived from this analysis. However we prefer to present less explicit but more general reasoning.*

References

- [Br] BRUK, V.M. *Of a class of boundary value problems with spectral parameter in the boundary condition.* Math. URSS Sbornik, Vol. **29**:2, 186 - 192.
- [Cal] CALKIN, J.W. *Abstract symmetric boundary conditions.* Trans. Amer. Math. Soc., Vol. **45** (1939), 369 - 442.
- [ChSt] CHU, L.J., STRATTON, J.A. *Elliptic and Spheroidal Wave Functions.* Journal of Math. and Phys., **20** (1941), 259 - 309. Reprinted in [SMCLC], p. 1 - 51.

- [DuSch] DUNFORD, N., SCHWARTZ, J.T. *Linear Operators. Part II. Spectral Theory. Self Adjoint Operators in Hilbert Space*. Intersc. Publ., Wiley and Sons. New York•London 1963.
- [Fl] FLAMMER, C. *Spheroidal Wave Functions*. Stanford University Press, Stanford, CA, 1957. ix+220.
- [GGK] GOHBERG, I., GOLDBERG, S., KAASHOEK, M.A. *Classes of Linear Operators. Vol. 1*. Birkhäuser, Basel•Boston•Berlin 1990.
- [HuPy] HUTSON, V.C.L., ПЫМ, J.S. *Application of Functional Analysis and Operator Theory*. Academic Press, New York•London 1980. xi+389 pp.
- [KaMa] KATSNELSON, V, MACHLUF, R. *The truncated Fourier operator*. II. arXiv:0901.2709.
- [KPS] КОМАРОВ, И.В., ПОНОМАРЕВ, Л.И., СЛАВЯНОВ, С.Ю. *Сфероидальные и Кулоновские Сфероидальные Функции*. Наука, Москва 1976. 319 сс. (In Russian.)
[KOMAROV, I.V., SLAVYANOV, S.YU., PONOMAREV, L.I. *Spheroidal and Coulomb spheroidal functions*. Nauka, Moscow 1976. 319 pp.]
- [Koch] KOCHUBEI, A.N. *Extensions of symmetric operators and symmetric binary relations*. Mat. Notes **17** (1975), 186 - 192.
- [Kr] КРЕЙН, М.Г. *Теория самосопряженных расширений полуограниченных эрмитовых операторов и ее приложения*. II. Матем. Сборник. Том **21**:3 (1947), 365 - 404 (Russian).
KREIN, M.G. *The theory of self-adjoint extensions of semibounded Hermitian operators and its applications*. II. Matem. Sbornik, Vol. **21**:3 (1947), 365 - 404.
- [LaP1] LANDAU, H., POLLAK, H.O. *Prolate spheroidal wave functions, Fourier analysis and uncertainty – II*. Bell System Techn. Journ. **40** (1961), 65 - 84.
- [LaP2] LANDAU, H., POLLAK, H.O. *Prolate spheroidal wave functions, Fourier analysis and uncertainty – III: The dimension of the space of essentially time- and band-limited signals*. Bell System Techn. Journ. **40** (1961), 1295 - 1336.
- [MSch] MEIXNER, J., SCHAFKE, F.W. *Mathieu'sche Funktionen und Spheroidfunktionen*. Springer-Verlag, Berlin•Gottingen•Heidelberg 1954.

- [Nai] NAIMARK, M.A. *Linear Differential Operators. Part II*. Frederic Ungar Publishing Co., New York 1968.
- [Neu] VON NEUMANN, J. *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*. Math. Ann., **102** (1929-1930), 49 - 131.
- [RoB] ROFE-BEKETOV, F.S. *Selfadjoint extensions of differential operators in a space of vector functions*. Soviet Math. Dokl., **10**:1 (1969), 188 - 192.
- [Sl1] SLEPIAN, D. *Prolated spheroidal wave functions, Fourier analysis and uncertainty – IV: Extension to many dimensions; generalized prolate spheroidal functions*. Bell System Techn. Journ. **43** (1964), 3009 - 3057.
- [Sl2] SLEPIAN, D. *On bandwidth*. Proc. IEEE **64**:3 (1976), 292–300.
- [Sl3] SLEPIAN, D. *Some comments on Fourier analysis, uncertainty and modelling*. SIAM Review, **25**:3, 1983, 379 - 393.
- [SlPo] SLEPIAN, D., POLLAK, H.O. *Prolated spheroidal wave functions, Fourier analysis and uncertainty – I*. Bell System Techn. Journ. **40** (1961), 43 - 63.
- [Sm] SMIRNOV, V.I. *A Course of Higher Mathematics*. Vol. 3, Part 2. Addison-Wesley, Reading MA • London 1964. X+700 pp.
- [St] STONE, M.H. *Linear transformation in Hilbert space and their Applications to Analysis*. (Amer. Math. Soc. Colloquim Publ., **15**). Amer. Math.Soc., New York, 1932.
- [SMCLC] STRATTON, J.A., MORSE, P.M., CHU, L.J., LITTLE J.D.C. CORBATÓ, F.J. *Spheroidal Wave Functions, including Tables*. MIT Press and Wiley, 1956. xi+300.

Victor Katsnelson
 Department of Mathematics
 The Weizmann Institute
 Rehovot, 76100, Israel
 e-mail:
 victor.katsnelson@weizmann.ac.il

Ronny Machluf
Department of Mathematics
The Weizmann Institute
Rehovot, 76100, Israel
e-mail:
`ronny-haim.machluf@weizmann.ac.il`